

Holography and entropy bounds in the plane wave matrix model

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ABSTRACT: As a quantum theory of gravity, Matrix theory should provide a realization of the holographic principle, in the sense that a holographic theory should contain one binary degree of freedom per Planck area. We present evidence that Bekenstein's entropy bound, which is related to area differences, is manifest in the plane wave matrix model. If holography is implemented in this way, we predict crossover behavior at strong coupling when the energy exceeds N^2 in units of the mass scale.

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1. Introduction

The holographic principle [1–3] requires that the surprising entropy bounds apparent in nature [4] should be manifest in quantum gravity. For asymptotically Anti-de Sitter spacetimes this was realized by the Maldacena conjecture [5] in which A binary degrees of freedom of an $SU(N)$ Yang-Mills theory suffice to describe a region of surface area A [6], in Planck units. This holographic behavior follows from a UV/IR relation that places a UV cutoff on the field theory as the size of bulk regions is decreased [7].

When the bulk region becomes as small as the AdS curvature scale, the field theory reaches its lowest possible UV cutoff. Only the zero-modes on the S^3 are retained, and the theory reduces to matrix quantum mechanics. At this point holography is still manifest with N^2 degrees of freedom describing a region of area N^2 . But for smaller regions it is not clear how to eliminate any further degrees of freedom from the field theory. For similar reasons holographic state counting is quite obscure in the matrix

models proposed to describe M-theory on asymptotically flat space [8] or on the plane wave in eleven dimensions [9].

In this paper, we propose that the entropy bound that becomes manifest in matrix quantum mechanics is not the covariant bound relating entropy to surface area. Rather, it is the Bekenstein bound [10], which states that the entropy of a system will not exceed its mass times its linear size. We find evidence that this bound is saturated by the matrix degrees of freedom available for the description of weakly curved geometries.

The Bekenstein bound appears to be connected to the holographic principle in that it arises from the generalized covariant entropy bound [11] in a certain weak-gravity limit [12]. In its regime of validity it is much tighter than the covariant bound. Thus, it may govern the emergence of local weakly gravitating regions from a more fundamental theory [13], similar to the more general role envisaged for the covariant bound. However, it has suffered from ambiguities in its definition of entropy (recent discussions of this problem include Refs. [14–20]).

Attempts to rectify this situation led to the proposal [21] to formulate Bekenstein’s bound in terms of the discrete light-cone quantization (DLCQ) of theories on backgrounds with a null Killing vector: The entropy in a sector with N units of momentum should not exceed $2\pi^2 N$:¹

$$S \leq 2\pi^2 N . \tag{1.1}$$

It is natural to test this proposal in the context of M-theory. M-theory is *defined* in terms of its DLCQ as a $U(N)$ matrix model [24]. Moreover, we expect that it represents a consistent quantum theory of gravity. This means that violations, if found, cannot be ascribed to an artificial choice of Lagrangian.

In particular, we shall study M-theory on the eleven-dimensional plane wave background. Unlike Matrix theory in flat space [8], the spectrum of the plane wave matrix model is discrete. It is known exactly [25] at large boosts μ , where the quantum mechanics is weakly coupled. However, the curvature of the gravity dual is strong in this regime, and the Bekenstein bound is not expected to apply.

At small μ , when the plane wave is weakly curved, the matrix model is strongly coupled. However, there are still infinite towers of protected states. Without a cutoff

¹In Ref. [12] the Bekenstein bound is derived as the integrated loss of cross-sectional area of parallel light-rays focused by a matter system (see also Ref. [22, 23]). It arises not in its original form ($S \leq 2\pi MR$, where R is the largest linear dimension), but in the stronger form $S \leq \pi M \Delta x$, where Δx is the width of the system along an arbitrary direction in its rest frame. It is more natural to work in the lightcone frame picked out by the light-sheet. One then has $S \leq \pi P_- \Delta x^-$, where the null coordinate x^- and the longitudinal momentum P_- are defined as usual [21]. Ambiguities in defining the spatial extent of quantum states can be suppressed by compactifying the null direction. Then the momentum is quantized ($P_- = 2\pi N / \Delta x^-$), and the bound becomes $S \leq 2\pi^2 N$. Here S is the logarithm of the number of discrete states [16] in the sector with N units of momentum.

on the lightcone energy the protected states alone would seem to contribute an infinite amount of entropy at any value of N , apparently violating the Bekenstein bound.

However, both the full spectrum at weak coupling and the spectrum of protected states at arbitrary coupling undergo crossover behavior when the energy in units of μ becomes of order N^2 , i.e., for

$$E_{\text{cross}} \sim N^2 . \quad (1.2)$$

The entropy of the full spectrum (at weak coupling) behaves as

$$S \sim E \quad (E \lesssim N^2) ; \quad S \sim N^2 \log \frac{E}{N^2} \quad (E \gtrsim N^2) . \quad (1.3)$$

The entropy of the protected states behaves as

$$S_{\text{p}} \sim (EN)^{1/3} \quad (E \lesssim N^2) ; \quad S_{\text{p}} \sim N \log \frac{E}{N^2} \quad (E \gtrsim N^2) . \quad (1.4)$$

At strong coupling, where the Bekenstein bound should apply, only the protected states are guaranteed to be present. We shall assume that they dominate the entropy in the microcanonical ensemble, at least in the vicinity of the crossover energy. Then the above results show that the system undergoes a kind of phase transition (smoothed out by finite N) at $E \sim N^2$, both at strong and at weak coupling. On the gravity side, one expects this type of behavior to be associated with a nonperturbative modification of the background [26] analogous to the Hawking-Page transition in Anti-de Sitter space [27, 28].

It would be interesting to develop a concrete proposal for the gravity interpretation of this transition in the 11D plane wave. No black hole solutions asymptotic to the 11D plane wave have been found, and it is possible that the eleven-dimensional spacetime interpretation breaks down entirely at the crossover energy.

States associated with a strong perturbation of the background do not contribute to the Bekenstein bound. Thus, under the stated assumptions, the entropy entering the bound is at most the crossover entropy,

$$S_{\text{p,cross}} \sim N . \quad (1.5)$$

Comparison with Eq. (1.1) finds the Bekenstein bound saturated, but not exceeded.

This paper is organized as follows. In Sec. 2 we consider the plane wave matrix model at weak coupling. Adapting general arguments for a Hagedorn/deconfinement phase transition in weakly coupled Yang-Mills theory [24, 26, 29–36], we find crossover behavior at $E \sim N^2$, Eq. (1.3). At this point the energy becomes large enough to excite all matrix degrees of freedom; the finite, quantum-mechanical nature of the finite N system is revealed, and is reflected in a much slower growth of entropy.

Because this result applies in a regime of strong curvature, it is not directly relevant to the Bekenstein bound. However, it lends additional support to the evidence we find in Sec. 3 for a similar transition at strong coupling. We consider the spectrum of protected multiplets. It receives no corrections at strong coupling when the model is conjectured to describe a weakly curved background. We find that it also undergoes a transition at $E \sim N^2$, as shown in Eq. (1.4).

Our results are suggestive but not conclusive. In Sec. 4 we clarify the additional assumptions required to interpret the crossover entropy $S_{\text{p,cross}} \sim N$ of protected states as a manifestation of the Bekenstein bound. In particular, one must assume that the crossover seen for protected states is representative of the full theory at strong coupling.

In Appendix A we discuss a subtlety arising for the center of mass degrees of freedom. Appendix B summarizes properties of protected states.

2. Entropy and crossover in the free theory

In this section we introduce the plane wave matrix model and discuss the crossover behavior of the entropy at energy N^2 at weak coupling when the full spectrum is known exactly.

2.1 The plane wave matrix model

The $U(N)$ plane wave matrix model is given by the Hamiltonian

$$\begin{aligned} H = R \operatorname{Tr} & \left(\frac{1}{2} \Pi_A^2 - \frac{1}{4} [X_A, X_B]^2 - \frac{1}{2} \Psi^\top \gamma^A [X_A, \Psi] \right) \\ & + \frac{R}{2} \operatorname{Tr} \left(\left(\frac{\mu}{3R} \right)^2 X_i^2 + \left(\frac{\mu}{6R} \right)^2 X_a^2 \right. \\ & \quad \left. + i \frac{\mu}{4R} \Psi^\top \gamma^{123} \Psi + i \frac{2\mu}{3R} \epsilon^{ijk} X_i X_j X_k \right) . \end{aligned} \quad (2.1)$$

Indices $A \dots$ run from 1 to 9; $i \dots$ run from 1 to 3; and $a \dots$ run from 4 to 9. This model was proposed [9] to describe M-theory on the maximally supersymmetric plane wave background of eleven-dimensional supergravity,

$$ds^2 = -2dx^+ dx^- + (dx^A)^2 - \left(\frac{\mu^2}{9} x^i x_i + \frac{\mu^2}{36} x^a x_a \right) (dx^+)^2 , \quad (2.2)$$

$$F_{123+} = \mu . \quad (2.3)$$

The parameter μ in the metric is a coordinate artifact; it can be set to any value by rescaling x^\pm . The M-theory limit of the matrix model (2.1) is obtained by taking

$N \rightarrow \infty$ while holding N/R fixed. In this limit the model must also become independent of μ .

At finite N the matrix model is expected to describe the DLCQ of M-theory, in the sector with N units of longitudinal momentum [24]. In this case, the coordinate x^- is periodically identified with period $2\pi R$. For finite N the boost-invariant quantity μ/R is a physical parameter that distinguishes qualitatively different regimes.

To see this, it is useful to think of μ as the curvature radius of the transverse dimensions as measured in the frame in which the periodically identified hypersurfaces have spatial distance R . To obtain a good geometric description we require both $\mu \ll 1$ and $R \gg 1$; hence, $\mu/R \ll 1$. For large μ/R the matrix model does not correspond to a classical background in any frame; in particular, we do not expect the Bekenstein bound to apply.

In the limit $\mu \rightarrow \infty$ at fixed N and R the plane wave matrix model (2.1) becomes free. For each partition of N there is a superselection sector with its own 1/2 BPS vacuum, corresponding to a collection of concentric fuzzy spheres [9]. Let us focus on the $X = 0$ sector given by the trivial partition $N = 1 + 1 + \dots + 1$. As we shall see, it exhibits the most rapid growth of entropy.

2.2 $X = 0$ sector

The $X = 0$ sector has Hamiltonian

$$H = \frac{\mu}{3} \sum_i A_i^\dagger A_i + \frac{\mu}{6} \sum_a A_a^\dagger A_a + \frac{\mu}{4} \sum_{I\alpha} \psi_{I\alpha}^\dagger \psi_{I\alpha} \quad (2.4)$$

and contains rank N matrix creation operators

$$A_i^\dagger = \sqrt{\frac{\mu}{6R}} X^i - i \sqrt{\frac{3R}{2\mu}} \Pi^i, \quad A_a^\dagger = \sqrt{\frac{\mu}{12R}} X^a - i \sqrt{\frac{3R}{\mu}} \Pi^a, \quad (2.5)$$

as well as fermionic operators $\psi_{I\alpha}$ [25].

Each creation operator contributes of order μ to the lightcone energy, so the dimensionless energy $E \equiv -P_+/\mu$ measures the number of quanta. Physical states must be gauge-invariant. Hence, they correspond to products of traces of products of creation operators.

It is important that there is more than one kind of matrix operator. There is one operator for each transverse direction that the system can move in, so their number, q , is of order ten. In general matrices do not commute, so a state with E quanta can

be made in q^E different ways² leading to an entropy $S \sim E \log q$. Since q is not very large, one has [26, 29]

$$S \sim E . \quad (2.6)$$

This Hagedorn behavior cannot persist indefinitely. At sufficiently high energy the entropy is dictated by the thermodynamics of a quantum mechanical system with about N^2 degrees of freedom:

$$S \sim N^2 \log \frac{E}{N^2} . \quad (2.7)$$

At a crossover energy of order N^2 the two expressions for the entropy match.

The argument for Hagedorn growth, Eq. (2.6), breaks down because trace relations can lead to identifications between states [26]. Apparently³ this effect becomes important only at $E \sim N^2$.

2.3 Full spectrum

Other sectors exhibit a less rapid growth of states. Consider the “irreducible” vacuum, which corresponds to the partition of N into just one term. It describes a single fuzzy membrane of momentum N/R . At small energies the entropy scales as $S_{M2} \sim E^{2/3}$, as it should for a 2+1 dimensional object. Thus, it grows more slowly than (2.6) below. Matching to the asymptotic density of states, Eq. (2.7), reveals that the crossover occurs at energies of order N^3 .

The differing crossover energies are easily understood as follows. Crossover happens when we notice that the matrix is finite, i.e., for energies large enough to excite all matrix degrees of freedom. In the trivial vacuum there are approximately qN^2 matrix elements. Each corresponds to a creation operator that increases E by about 1. Thus,

²This is a slight simplification, even leaving aside the issue of trace relations discussed below. At least one trace must be taken, and of course multiple traces are allowed as well. Thus, we have undercounted states since the sequence of E operators can be sprinkled with traces. However, this cannot multiply the number of states by more than the number of partitions of E , $p(E) \approx \exp(\sqrt{\frac{2}{3}}\pi\sqrt{E})$. Hence it will correct the entropy (2.6) at most by a subleading term of order \sqrt{E} . We have also overcounted states because states related by cyclic exchange of the creation operators within a trace are not independent. This will reduce the entropy at most by a term of order $\log E$, which again is negligible compared to Eq. (2.6).

³There is tentative direct evidence for this. One can show that all single-trace states of length up to N are independent, even for $q > 1$ [37]. For $q = 1$ all traces longer than N decompose into products of traces. Hence, for $q > 1$ trace relations also set in at lengths exceeding N , at least for some traces. In a typical partition of E most traces have length of order \sqrt{E} , which will exceed N once E becomes larger than N^2 . Hence, this is the point at which we expect a typical state in the Hagedorn spectrum to become identified with other states by trace relations. This argument assumes that trace relations are not dominated by relations involving products of more than one trace on both sides of the equality.

The degeneracy of each multiplet is of order n^4 . The entire multiplet can be obtained by acting with supersymmetry generators on its primary states, which are associated with the $SO(6)$ transverse directions X^a (see Appendix B).

3.2 $X = 0$ sector

We will first discuss the entropy of protected states in individual superselection sectors of the free theory ($\mu \rightarrow \infty$). As explained in Appendix A, we suppress the center of mass degrees of freedom. Then the single-membrane sector contains neither operators nor states of weight $(0, n, 0)$.

Let us focus instead on the $X = 0$ sector, given in Eqs. (2.4) and (2.5). The primaries for $(0, n, 0)$ representations are of the form

$$S^{a_1 \dots a_n} \text{Tr}_N (A_{a_1}^\dagger \dots A_{a_n}^\dagger) |0\rangle . \quad (3.4)$$

where $S^{a_1 \dots a_n}$ is a totally symmetric traceless tensor of $SO(6)$, and we use the notation Tr_N to indicate that an arbitrary number of traces may be sprinkled among the n operators, subject to the constraint that each trace contain at most N operators. (Because of the complete symmetrization, traces longer than N are guaranteed not to be independent. In order to exclude the $U(1)$ one should also require that each trace include at least two operators, but this will give a negligible correction to the entropy.)

Ordering the traces by their length, we may represent each primary by a Young diagram with n boxes and at most N rows, where each column represents one trace. These auxiliary diagrams do not indicate a group representation; they simply keep track of all the different ways one can *produce* the $(0, n, 0)$ representation, and thus of its degeneracy in the spectrum.

Hence, the number of protected representations of energy E is given by the number of restricted partitions of E ,

$$p_N(E) \approx \frac{1}{4\sqrt{3}E} \exp \left[\pi \sqrt{\frac{2E}{3}} - \frac{\sqrt{6E}}{\pi} \exp \left(\frac{-\pi N}{\sqrt{6E}} \right) \right] . \quad (3.5)$$

This formula is valid for $1 \ll E \ll N^2$. The entropy of protected states will thus grow as

$$S_{\text{p } (X=0)} \sim 2\pi \sqrt{\frac{E}{6}} \quad (3.6)$$

until a crossover threshold around $E \sim N^2$, when

$$S_{\text{p,cross } (X=0)} \sim N . \quad (3.7)$$

The contribution $4 \log E$ from the degeneracy of states within each protected multiplet is negligible.

One can understand this result from a number of perspectives. Naively, one might expect crossover behavior for $p_N(E)$ around $E = N$ when the restriction to at most N rows first becomes nontrivial. However, what matters for the entropy is how the restriction acts on *typical* partitions. For large E the vast majority of unrestricted partitions contain about \sqrt{E} rows ranging in length up to \sqrt{E} ; that is, the Young diagram associated to a typical partition looks very roughly like a triangle of height and width \sqrt{E} .⁴ This means that the restriction to at most N rows will not become important until $E \sim N^2$, when a typical partition would prefer to have more than N rows. Therefore, we expect crossover behavior in the number of states for $E \sim N^2$, not $E \sim N$.

Let us gain more insight into the behavior of the entropy beyond the crossover. The partition function for protected representations is identical to that for N bosons in a harmonic oscillator or for a system of N harmonic oscillators of frequency $1, 2, \dots, N$, with zero point energies removed:

$$Z(T) = \prod_{j=1}^N \frac{1}{1 - e^{-j/T}} . \quad (3.8)$$

Computing the entropy and energy from $Z(T)$ for $1 \ll T \ll N$ yields Eq. (3.5) [40].

For $T \rightarrow \infty$ this partition function yields $S_{\text{p}(X=0)} = 2N + N \log(T/N)$ and $E \sim NT$, consistent with the thermodynamics of a quantum mechanical system with N degrees of freedom. Hence

$$S_{\text{p}(X=0)} = 2N + N \log \frac{E}{N^2} . \quad (3.9)$$

The onset of this asymptotic behavior is for $T = N$, when the temperature is large enough to excite the frequency N oscillator. This corresponds to a lower bound $E \sim N^2$ for the asymptotic regime, consistent with the upper limit obtained for the $T < N$ regime. The crossover entropy, $S_{\text{p,cross}} \sim N$, is also consistent with Eq. (3.7).

3.3 Full protected spectrum

At any finite coupling, and in particular at strong coupling, all sectors of the free theory must be included in the microcanonical ensemble. So far we have considered only the

⁴More precisely, by computing expected occupation numbers of $N \rightarrow \infty$ oscillators in Eq. (3.8) below one obtains the limiting curve $e^{x/T} + e^{y/T} = 1$, where x and y are coordinates along the edges of the Young diagram, and $T = \sqrt{6E}/\pi$. This means that the expected height and width of the diagram is $T \log T$. Hence it will begin to exceed N slightly earlier than for $T \sim N$. This slows down the growth of the degeneracy but only enough to modify order one prefactors in the entropy. One still has $S \sim \sqrt{E}$ until $T \sim N$ [39].

protected states of the $X = 0$ vacuum ($N = 1 + \dots + 1$). The other vacua, corresponding to nontrivial partitions of N , will contribute additional protected states.

Consider a general partition of N , $(N_1^{M_1} \dots N_l^{M_l})$. In this notation M_i denotes how many times N_i appears in the sum, i.e., $N = \sum_{i=1}^l M_i N_i$. Each term $N_i^{M_i}$ corresponds to a stack of M_i coincident fuzzy spheres of individual momenta N_i/R . Each stack contributes a rank M_i protected matrix creation operator. Thus, for the purposes of protected states, the momentum of the stack is irrelevant; only its “height” M_i matters.

The counting problem is not only how E can be partitioned into traces, but at the same time how N can be partitioned into stacks, whose height controls the maximum length of traces and whose number controls the number of different types of traces that can be constructed.

To compute the total number of protected states it is convenient to introduce a chemical potential ν conjugate to N . Then the problem is an extension of the canonical partition function generating the partitions of E . A stack $N_i^{M_i}$ forms in response to the chemical potential ν , in analogy to an oscillator of frequency j being excited to occupation number m_j in response to a temperature $T = 1/\beta$.

The grand canonical partition function for a single stack of fuzzy spheres is thus

$$Z_k(\beta, \nu) = \sum_{N, E=0}^{\infty} e^{-\nu k N} e^{-\beta E} p_N(E) , \quad (3.10)$$

where k is the number of units of longitudinal momentum of each sphere in the stack. Using Eq. (3.8) this becomes [41]

$$Z_k(\beta, \nu) = \prod_{n=0}^{\infty} \frac{1}{1 - e^{-\nu k} e^{-\beta n}} . \quad (3.11)$$

A general vacuum contains stacks of fuzzy spheres of arbitrary momenta, described by the product of these partition functions (analogous to the product of partition functions of harmonic oscillators of different frequencies):

$$Z(\beta, \nu) = \prod_{k=1}^{\infty} Z_k(\beta, \nu) . \quad (3.12)$$

For small β and ν this evaluates to

$$\log Z(\beta, \nu) \approx \frac{c}{\beta \nu} , \quad (3.13)$$

with $c \approx 1.20206$.

The entropy is obtained from the integral transform

$$e^{S_p(N,E)} = \int d\nu d\beta e^{\nu N + \beta E} Z(\beta, \nu) , \quad (3.14)$$

where in the saddlepoint approximation

$$N = \frac{c}{\beta\nu^2} , \quad E = \frac{c}{\beta^2\nu} . \quad (3.15)$$

This yields a density of states $\frac{1}{2}(\frac{c}{N^2 E^2})^{1/3} \exp[3(cNE)^{1/3}]$, so that to leading order [41]

$$S_p(N, E) = 3(cNE)^{1/3} . \quad (3.16)$$

Using Eq. (3.15), the assumption of small β and ν translates into the regime of validity

$$\sqrt{N} \ll E \ll N^2 . \quad (3.17)$$

A lower crossover is at $E^2 \sim N$, when the entropy becomes completely dominated by the number of vacua (the partitions of N). We are interested in the higher crossover at $E \sim N^2$. At this point the entropy becomes dominated by the states in the $X = 0$ sector. From Eq. (3.16) we find again that the entropy at this crossover energy is of order N :

$$S_{p,\text{cross}} \sim N . \quad (3.18)$$

To approach the crossover from the other side ($E \gg N^2$), we evaluate the partition function (3.12) for $\nu \gg 1$:

$$\log Z(\beta, \nu) \approx \frac{1}{\beta e^\nu} , \quad (3.19)$$

with saddlepoint

$$E = \frac{1}{\beta^2 e^\nu} , \quad N = \frac{1}{\beta e^\nu} . \quad (3.20)$$

Hence the asymptotic behavior of the entropy at very high energies is

$$S_p = 2N + N \log \frac{E}{N^2} . \quad (3.21)$$

For $E \sim N^2$, this matches up with Eq. (3.16) at the crossover entropy (3.18).

4. Assumptions and implications

Our analysis shows that the Bekenstein bound, in the DLCQ form $S \lesssim N$, is satisfied and approximately saturated in the plane wave matrix model under the following assumptions:

1. The protected states correctly estimate the entropy going into the Bekenstein bound, at least for energies approaching the crossover scale $E = N^2$ from below.
2. For energies above the crossover the matrix model does not describe a weakly gravitating system.

In this section we discuss why these assumptions are needed, point out that the first lends plausibility to the second, and speculate about their implications for the matrix model.

Applicability of the Bekenstein bound requires a good semiclassical background. This is why we have studied the spectrum for small values of μ , where curvatures are small. Because the matrix model is strongly coupled there, we were only able to discuss protected states.

But a weak background is not enough. The states included in the Bekenstein bound must themselves be weakly gravitating, in the sense that they are incapable of focussing light significantly over a distance set by their own spatial size [12].

Some protected states, even in a weakly curved background, may well have large backreaction. (In the context of AdS, an example are single gravitons with energy above the Planck energy.) Thus, not all protected states necessarily contribute to the entropy in the Bekenstein bound. On the other hand, there may be some unprotected states (in the AdS analogy, say, multiple weak gravitons) which have small backreaction and do contribute.

Such effects will not be important if they modify the entropy only by factors of order one, or only far from the crossover energy. But if they are significant near $E \sim N^2$, they can affect one or both of our conclusions (that the Bekenstein bound is satisfied, and that it is saturated). This is why the first assumption is needed. Without the second assumption, the Bekenstein bound would be violated by states with $E \gg N^2$.

We are hopeful that the validity of both assumptions can be clarified thanks to recent progress in the physical understanding of the plane wave states and their geometry at strong coupling. Excitations of M5 branes appear in the spectrum of protected states [42], and their multiparticle states are likely to play a role in this analysis. Since the geometry near a single M5 brane is not smooth, it will be interesting to consider solutions with coincident M5 brane configurations [41] in this context. We leave this to future work.

If the first assumption is correct, it will have an interesting implication that makes the second assumption more plausible: The crossover behavior at $E \sim N^2$ (in units of μ), which is quite well understood at weak coupling, will persist at strong coupling. This may seem surprising: When $\mu \ll M_{\text{Pl}}$, why should the characteristic scale continue to be μ , rather than, say, the Planck scale? Consider a scattering problem with impact

parameter much shorter than the transverse curvature radius, $(R/\mu)^{1/2}$ (which is large in Planck units in the strong coupling regime). Such processes should be described as in the flat space matrix model. Therefore, μ should be dynamically irrelevant, and all interesting scales should arise from the Planck mass and powers of N .

However, this logic applies only to short timescales. In the flat space case ($\mu = 0$ exactly) two gravitons that scatter can move off to infinity uninhibited by the quartic interaction since off-diagonal excitations are frozen out at large separation. But gravitons that scatter in the plane wave will eventually come to notice that they live in a confining background and are really in a bound state. This takes a time of order $1/\mu$, which diverges in the $\mu \rightarrow 0$ limit. None of these bound states survive at $\mu = 0$, so we do not expect the BFSS limit to be smooth in this sense. This argument suggests that μ remains an important scale in the microcanonical ensemble at all nonzero values of μ no matter how small.

Phase transitions in Yang-Mills theory have been analyzed in detail in Ref. [26]. In terms of their classification, our analysis suggests that the plane wave matrix model will *not* behave like, say, ordinary $d = 4$ Yang-Mills theory, whose transition temperature can be set by two different scales. ($T_{\text{crit}} \sim 1/R$ for compactification on a small three-manifold of size R^3 , but $T \sim \Lambda_{\text{QCD}}$ for $R \gg 1/\Lambda_{\text{QCD}}$, when coupling is strong.) Rather the matrix model should behave like $d = 4$, $\mathcal{N} = 4$ super-Yang-Mills theory on a three-sphere, whose phase structure is expected to be the same at strong and weak coupling. (The critical temperature is of order $1/R$ and the crossover energy is N^2/R , where R is the size of the S^3 .)

It is interesting to ask about the gravity dual to the matrix model above the crossover energy. The deconfinement phase transition of $\mathcal{N} = 4$, $d = 4$ super-Yang-Mills theory is related via AdS/CFT to the Hawking-Page transition in AdS [28, 43]. More generally, it has been suggested that the high energy phase of a large- N Yang-Mills theory always corresponds to the presence of a black object in the dual closed string background [26]. This amounts to a nonperturbative modification of the background.

Here we are dealing with an M-theory background, but it is tempting to speculate that it becomes similarly modified. Conceivably, the bulk dual of the crossover could be more drastic: the gravity interpretation may break down entirely. Unlike for AdS, no black hole or black string solutions asymptotic to the eleven-dimensional plane wave are known. Moreover, it is doubtful that the canonical ensemble can be defined any more sensibly than it can for flat space.⁵ (To put flat space at a finite temperature requires nonzero constant energy density and thus infinite energy; in a theory with gravity this invalidates the background.)

⁵We thank M. Van Raamsdonk for stressing this point to us.

In AdS/CFT gravity turns off for $N \rightarrow \infty$, where the crossover becomes a sharp phase transition. In this limit there is no longer a good gravity description above the critical temperature. Unlike AdS, the M-theory limit on the plane wave *requires* $N \rightarrow \infty$. This is another reason to suspect that there may be no sensible eleven-dimensional gravity dual beyond the crossover energy.

For the purposes of obtaining a cutoff on the entropy entering the Bekenstein bound, it only matters that such states can no longer be described as small perturbations of the plane wave background. Any of the scenarios we have described—black hole formation, or a complete breakdown of the geometric interpretation—would guarantee this.

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A. The $N = 1$ sector

In the geometric regime the matrix model is strongly coupled, except in the case $N = 1$ when all interaction terms vanish. This sector is present for all values of N since $U(N) = U(1) \times SU(N)$. It describes the decoupled dynamics of the center of mass. In this appendix we discuss the spectrum of the $U(1)$ sector. We will argue that it does not contribute to the entropy in the Bekenstein bound because all of its states are gauge copies of each other under diffeomorphisms.

The Hamiltonian and creation operators for the $N = 1$ sector can be obtained from Eqs. (2.4) and (2.5) as a special case. It is the Hamiltonian for a particle in a nine-dimensional harmonic oscillator, namely a graviton of longitudinal momentum $1/R$ in linearized 11D supergravity [44]. Its frequency is set by the curvature parameter of the plane wave, μ , and its effective mass is given by the longitudinal momentum, $1/R$. Hence, the characteristic length scale of the oscillator is $(R/\mu)^{1/2}$. The spectrum is given by infinite towers of bosonic states generated by the creation operators A^\dagger , with an additional multiplicity of 2^8 from the fermionic operators ψ^\dagger .

Application of the Bekenstein bound requires not only that the background be weakly curved, but also that the backreaction of the system we study be negligible. In particular, the light-sheet in the x^- direction should not focus much over a distance

$2\pi R$. The total area loss of the light-sheet is N [12]. This corresponds to negligible focussing only if the transverse area of the system (here, the graviton) is much larger than N :

$$A \gg N . \tag{A.1}$$

In the ground state, the graviton occupies a transverse area of order $(R/\mu)^{9/2}$, so weak backreaction requires

$$\mu/R \ll 1 . \tag{A.2}$$

This condition is automatically satisfied in a weakly curved background; see Sec. 2.1.

The spread of the graviton wave function in the transverse directions will be of order $\sqrt{RE/\mu}$. The area loss along x^- is N independently of E , so the backreaction becomes weaker for excited states. The point is that the spreading of the wave function over a nine-dimensional area overcompensates for the larger energy leading to a decrease in energy density and in backreaction. Hence, the spectrum will not be truncated due to large backreaction.

Under the criteria offered so far, it would appear that all states of the nine-dimensional oscillator contribute to the entropy as long as we choose $\mu/R \ll 1$. Their number is infinite⁶, so the Bekenstein bound would seem to be violated already for $N = 1$.

However, this is clearly wrong for the simple reason that the states we have considered are all the same. It is easiest to see this in the (overcomplete) basis of coherent states. All coherent states can be mapped to the ground state by an $SU(4|2)$ transformation, i.e., by a symmetry transformation that leaves the form (2.3) of the plane wave metric invariant. The required generators [45] commute with ∂_- , so the (discrete) longitudinal momentum is not affected by this transformation. Hence, the Bekenstein bound is trivially satisfied for $N = 1$.

The same argument will apply to the center-of-mass motion for larger values of N . In the matrix model this corresponds to the $U(1)$ factor that decouples from the $SU(N)$ degrees of freedom. We discard the $U(1)$ states for the same reason that we would not count different boosts of the same system as distinct bound states in flat space.

B. Protected representations

Here we give a brief summary of the form and properties of supersymmetrically protected states in the plane wave matrix model. For more details see, e.g., Refs. [38,41,42].

⁶The entropy grows with energy as $S_{N=1}(E) \sim 9 \log E$, but there is no cutoff on E .

The symmetry algebra of the eleven-dimensional plane wave is a basic classical Lie superalgebra

$$SU(4|2) \supset SU(4) \times SU(2) \times U(1)_H \sim SO(6) \times SO(3) \times U(1)_H . \quad (B.1)$$

The bosonic subalgebra $SO(6) \times SO(3)$ describes the symmetry of the nine-dimensional transverse plane, and the Hamiltonian is the $U(1)_H$ hypercharge. One can also think of this symmetry group as arising in the Penrose limit of AdS/CFT. On the AdS side, the Penrose limit of $AdS_7 \times S^4$ gives the plane wave. On the CFT side, the corresponding contraction of the $\mathcal{N} = 4$ superconformal group $SU(2, 2|4)$ gives the supergroup $SU(4|2)$.

At $\mu \rightarrow \infty$, all superrepresentations in the plane wave matrix model are tensor representations. Hence they are described by Young tableaux. To distinguish a supertableau from a bosonic tableau, slashed boxes are used. Any $SU(4|2)$ superrepresentation can be completely decomposed into representations of the bosonic subgroup $SU(4) \times SU(2)$. For example, the superrepresentation that is the sole building block of the entire $U(1)$ spectrum has the following decomposition

$$\boxed{\diagup} = (\boxed{}, 1) \oplus (\boxed{}, \boxed{}) \oplus (1, \boxed{}) . \quad (B.2)$$

Conversely, starting with a highest weight bosonic representations $|\psi\rangle$, the full superrepresentation can be recovered by acting with the 2^8 combinations of fermionic lowering operators. If the resulting states are all independent, the superrepresentation is called “typical”; otherwise, “atypical”.

A special set of multiplets are nonperturbatively protected from receiving corrections to their energy as the parameter μ of the matrix model is modified. The Young supertableaux of these “doubly atypical” representations have two rows of equal length

$$\cdot, \boxed{\diagup}, \boxed{\diagup\diagup}, \dots, \underbrace{\boxed{\diagup\diagup\diagup\diagup\diagup\diagup\diagup\diagup}}_n, \dots \quad (B.3)$$

and dimension

$$\dim_{DA}(n) = \frac{1}{3} (4n^4 + 16n^3 + 20n^2 + 8n + 3) . \quad (B.4)$$

Their bosonic decomposition is (for $n \geq 2$)

$$\begin{aligned} \underbrace{\boxed{\diagup\diagup\diagup\diagup\diagup\diagup\diagup\diagup}}_n &= \left(\underbrace{\boxed{}}_n, 1 \right)_{\frac{1}{4} \text{ BPS}} \oplus \left(\underbrace{\boxed{}}_n, \boxed{} \right)_{\frac{1}{8} \text{ BPS}} \oplus \left(\underbrace{\boxed{}}_{n-1}, \boxed{} \right)_{\frac{1}{8} \text{ BPS}} \\ &\oplus \left(\underbrace{\boxed{}}_n, \boxed{} \right)_{\frac{1}{4} \text{ BPS}} \oplus \left(\underbrace{\boxed{}}_{n-1}, \boxed{} \right)_{\frac{1}{8} \text{ BPS}} \oplus \left(\underbrace{\boxed{}}_{n-2}, \boxed{} \right)_{\frac{1}{4} \text{ BPS}} , \quad (B.5) \end{aligned}$$

where, following Ref. [38], each bosonic subrepresentations has been labeled by the fraction of the 32 supersymmetries it preserves. The energy of all constituent states is of order $E \sim n$.

The nonstandard supersymmetry algebra on the plane wave allows for BPS states with nonzero energy. This follows from the commutation relations between the supersymmetry generators and the Hamiltonian H . Schematically,

$$\begin{aligned}\{Q^\dagger, Q\} &\sim H - \mu M^{ij} - \mu M^{ab} , \\ [H, Q] &\sim \mu Q ,\end{aligned}\tag{B.6}$$

where the $SO(3)$, $SO(6)$ rotation generators M^{ij} , M^{ab} allow for positive definite H when $\{Q^\dagger, Q\} = 0$.

References

- [1] G. 't Hooft: *Dimensional reduction in quantum gravity*, gr-qc/9310026.
- [2] L. Susskind: *The world as a hologram*. J. Math. Phys. **36**, 6377 (1995), hep-th/9409089.
- [3] R. Bousso: *Holography in general space-times*. JHEP **06**, 028 (1999), hep-th/9906022.
- [4] R. Bousso: *A covariant entropy conjecture*. JHEP **07**, 004 (1999), hep-th/9905177.
- [5] J. Maldacena: *The large N limit of superconformal field theories and supergravity*. Adv. Theor. Math. Phys. **2**, 231 (1998), hep-th/9711200.
- [6] L. Susskind and E. Witten: *The holographic bound in Anti-de Sitter space*, hep-th/9805114.
- [7] A. W. Peet and J. Polchinski: *UV/IR relations in AdS dynamics*. Phys. Rev. D **59**, 065011 (1999), hep-th/9809022.
- [8] T. Banks, W. Fischler, S. H. Shenker and L. Susskind: *M theory as a matrix model: A conjecture*. Phys. Rev. D **55**, 5112 (1997), hep-th/9610043.
- [9] D. Berenstein, J. M. Maldacena and H. Nastase: *Strings in flat space and pp-waves from $N = 4$ super Yang Mills*. JHEP **04**, 013 (2002), hep-th/0202021.
- [10] J. D. Bekenstein: *A universal upper bound on the entropy to energy ratio for bounded systems*. Phys. Rev. D **23**, 287 (1981).
- [11] E. E. Flanagan, D. Marolf and R. M. Wald: *Proof of classical versions of the Bousso entropy bound and of the Generalized Second Law*. Phys. Rev. D **62**, 084035 (2000), hep-th/9908070.

- [12] R. Bousso: *Light-sheets and Bekenstein's bound*. Phys. Rev. Lett. **90**, 121302 (2003), hep-th/0210295.
- [13] R. Bousso: *Flat space physics from holography*. JHEP **05**, 050 (2004), hep-th/0402058.
- [14] D. N. Page: *Defining entropy bounds* (2000), hep-th/0007238.
- [15] J. D. Bekenstein: *On Page's examples challenging the entropy bound* (2000), gr-qc/0006003.
- [16] R. Bousso: *Bound states and the Bekenstein bound*. JHEP **02**, 025 (2004), hep-th/0310148.
- [17] D. Marolf and R. Roiban: *Note on bound states and the Bekenstein bound*. JHEP **08**, 033 (2004), hep-th/0406037.
- [18] D. Marolf, D. Minic and S. F. Ross: *Notes on spacetime thermodynamics and the observer-dependence of entropy*. Phys. Rev. **D69**, 064006 (2004), hep-th/0310022.
- [19] D. N. Page: *Hawking radiation and black hole thermodynamics*. New J. Phys. **7**, 203 (2005), hep-th/0409024.
- [20] J. D. Bekenstein: *How does the entropy/information bound work ?* (2004), quant-ph/0404042.
- [21] R. Bousso: *Harmonic resolution as a holographic quantum number*. JHEP **03**, 054 (2004), hep-th/0310223.
- [22] R. Bousso, E. E. Flanagan and D. Marolf: *Simple sufficient conditions for the generalized covariant entropy bound*. Phys. Rev. D **68**, 064001 (2003), hep-th/0305149.
- [23] A. Strominger and D. M. Thompson: *A quantum Bousso bound*. Phys. Rev. **D70**, 044007 (2004), hep-th/0303067.
- [24] L. Susskind: *Matrix theory black holes and the Gross-Witten transition* (1997), hep-th/9805115.
- [25] K. Dasgupta, M. M. Sheikh-Jabbari and M. Van Raamsdonk: *Matrix perturbation theory for M-theory on a pp-wave*. JHEP **05**, 056 (2002), hep-th/0205185.
- [26] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk: *The Hagedorn / deconfinement phase transition in weakly coupled large N gauge theories*. Adv. Theor. Math. Phys. **8**, 603 (2004), hep-th/0310285.
- [27] S. W. Hawking and D. N. Page: *Thermodynamics of black holes in Anti-de Sitter space*. Commun. Math. Phys. **87**, 577 (1983).

- [28] E. Witten: *Anti-de Sitter space and holography*. Adv. Theor. Math. Phys. **2**, 253 (1998), hep-th/9802150.
- [29] B. Sundborg: *The Hagedorn transition, deconfinement and $N = 4$ sym theory*. Nucl. Phys. **B573**, 349 (2000), hep-th/9908001.
- [30] O. Aharony, J. Marsano, S. Minwalla and T. Wiseman: *Black hole - black string phase transitions in thermal 1+1 dimensional supersymmetric Yang-Mills theory on a circle*. Class. Quant. Grav. **21**, 5169 (2004), hep-th/0406210.
- [31] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas and M. Van Raamsdonk: *A first order deconfinement transition in large N Yang- Mills theory on a small S^{*3}* . Phys. Rev. **D71**, 125018 (2005), hep-th/0502149.
- [32] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, M. Van Raamsdonk and T. Wiseman: *The phase structure of low dimensional large N gauge theories on tori* (2005), hep-th/0508077.
- [33] M. Li, E. J. Martinec and V. Sahakian: *Black holes and the SYM phase diagram*. Phys. Rev. **D59**, 044035 (1999), hep-th/9809061.
- [34] S. Hadizadeh, B. Ramadanovic, G. W. Semenoff and D. Young: *Free energy and phase transition of the matrix model on a plane-wave*. Phys. Rev. **D71**, 065016 (2005), hep-th/0409318.
- [35] G. W. Semenoff: *Matrix model thermodynamics* (2004), hep-th/0405107.
- [36] K. Furuuchi, E. Schreiber and G. W. Semenoff: *Five-brane thermodynamics from the matrix model* (2003), hep-th/0310286.
- [37] R. Bousso and A. L. Mints: *Decoding the matrix: Coincident membranes on the plane wave* (2005), hep-th/0510121.
- [38] K. Dasgupta, M. M. Sheikh-Jabbari and M. Van Raamsdonk: *Protected multiplets of M -theory on a plane wave*. JHEP **09**, 021 (2002), hep-th/0207050.
- [39] V. Balasubramanian, J. de Boer, V. Jejjala and J. Simon: *The library of Babel: On the origin of gravitational thermodynamics* (2005), hep-th/0508023.
- [40] M. N. Tran, M. V. N. Murthy and R. K. Bhaduri: *On the quantum density of states and partitioning an integer*. Annals Phys. **311**, 204 (2004), math-ph/0309020.
- [41] H. Lin and J. Maldacena: *Fivebranes from gauge theory* (2005), hep-th/0509235.
- [42] J. Maldacena, M. M. Sheikh-Jabbari and M. Van Raamsdonk: *Transverse fivebranes in matrix theory*. JHEP **01**, 038 (2003), hep-th/0211139.

- [43] E. Witten: *Anti-de Sitter space, thermal phase transition, and confinement in gauge theories*. Adv. Theor. Math. Phys. **2**, 505 (1998), hep-th/9803131.
- [44] T. Kimura and K. Yoshida: *Spectrum of eleven-dimensional supergravity on a pp-wave background*. Phys. Rev. **D68**, 125007 (2003), hep-th/0307193.
- [45] M. Blau and M. O’Loughlin: *Homogeneous plane waves*. Nucl. Phys. **B654**, 135 (2003), hep-th/0212135.